General relativity

Cosmology Block Course 2013

Markus Pössel & Björn Malte Schäfer

Haus der Astronomie und Astronomisches Recheninstitut

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Contents

The motivation for Einstein's general relativity 1

The basics of (classical) mechanics:

- Natural, inertial motion: straight lines, constant speed
- Deviation from this natural motion due to *forces* ($\vec{F} = m\ddot{\vec{x}}$)
- Additional models describe force properties (e.g. in terms of charges)

Example: Newton's gravitational force

$$\vec{F}(\vec{r}) = -GMm\,\frac{\vec{r}-\vec{r_q}}{|\vec{r}-\vec{r_q}|^3}.$$

How to identify inertial motion?

- How to separate real from inertial forces (e.g. rotating coordinate system: centrifugal or Coriolis force?)
- Case-by-case: Identify all real forces separately
  $\Rightarrow$ no good as a fundamental definition!
- What do all the inertial forces have in common?
  $\Rightarrow$ they're really accelerations!
- Decompose

$$\ddot{\vec{x}} = \frac{1}{m}\vec{F}(\vec{x}, \dot{\vec{x}}, t) - \vec{A}(t) - 2\vec{\omega} \times \dot{\vec{x}} - \vec{\omega} \times (\vec{\omega} \times \vec{x}) - \dot{\vec{\omega}} \times \vec{x}$$

...but there's a catch.

$$\ddot{\vec{x}} = \frac{1}{m}\vec{F}(\vec{x},\dot{\vec{x}},t) - \vec{A}(t) - 2\vec{\omega}\times\dot{\vec{x}} - \vec{\omega}\times(\vec{\omega}\times\vec{x}) - \dot{\vec{\omega}}\times\vec{x}$$

Can we use test particles to separate the different contributions?

E.g. to tease out contribution of the electromagnetic force: use test particles with different specific charge

$$\frac{q}{m}.$$

Problem: gravity!

Why gravity is special

$$\vec{F}(\vec{r}) = -GMm\,\frac{\vec{r}-\vec{r_q}}{|\vec{r}-\vec{r_q}|^3}.$$

Acceleration of test particle in the field of a large mass M:

$$\frac{1}{m}\vec{F}(\vec{r}) = -GM\,\frac{\vec{r}-\vec{r_q}}{|\vec{r}-\vec{r_q}|^3}.$$

Acceleration is independent of particle properties!

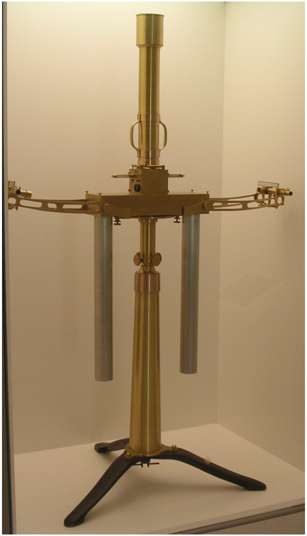
The two roles of mass

Inertial mass m_i vs. gravitational mass (charge) m_g:

$$\vec{F} = m_i \ddot{\vec{x}} \quad \text{vs.} \quad \vec{F}(\vec{r}) = -GMm_g \, \frac{\vec{r} - \vec{r_q}}{|\vec{r} - \vec{r_q}|^3}$$

...and apparently, $m_g = m_i$.

Right: Replica of Eötvös's torsion balance (at the Berlin Einstein Exhibition 2005)



A new natural motion

Consequence of not being able to disentangle *gravitational acceleration* $\vec{g}(\vec{x},t)$ and *inertial acceleration* $\vec{A}(t)$ completely:

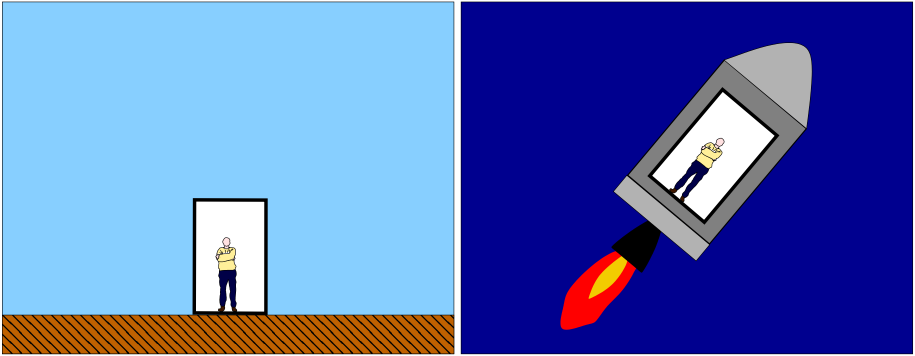
Replace

Test particle motion = inertial motion + deflection by forces

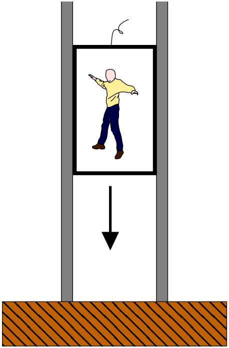
by

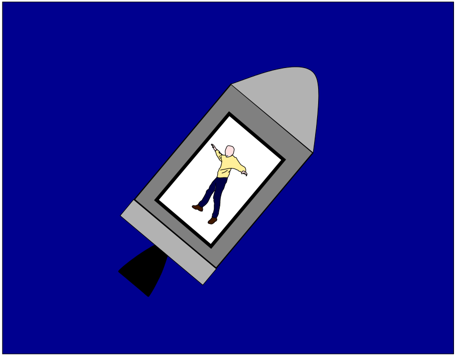
Test particle motion = free fall + deflection by forces

Is gravity the same as acceleration?



Is gravity the same as acceleration?





Microgravity in free fall

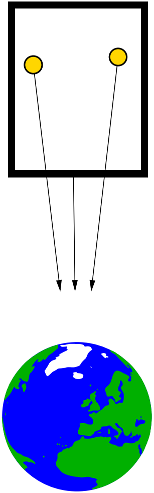
Capsule in the Glenn Research Center drop tower.

Image: NASA/GRC/P. Riedel, A. Lukas

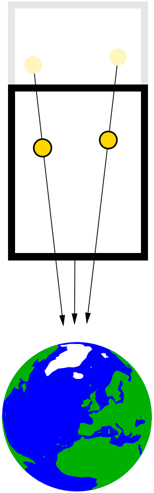
Microgravity in free fall = orbit

Chris Hadfield w/water bubble aboard the ISS. Image: NASA

Is there really no gravity in free fall?

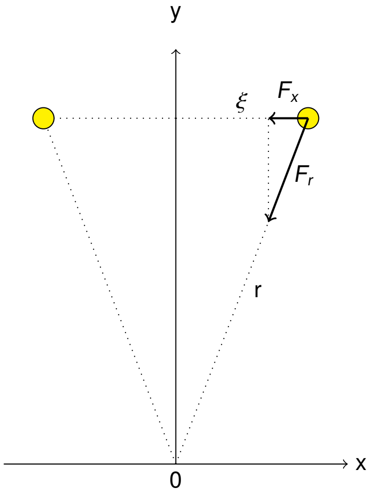


Is there really no gravity in free fall?



tidal forces!

Tidal forces



$$F_x = \frac{\xi}{r} \cdot F_r$$

$$\Rightarrow F_x = -\frac{GMm}{r^3} \cdot \xi$$

Tidal force strength falls off faster than $1/r^2$!

The limits of free fall

Typical in our example: tidal force proportional to separation, $F \sim \xi$

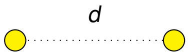
Wait sufficiently long, and you will see effects even for small ξ.

Principle of equivalence

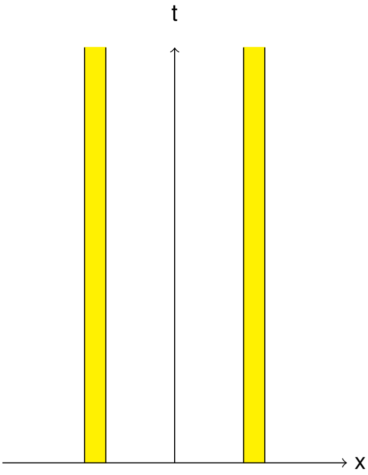
In an infinitesimally small region of space-time, the laws of physics, as measured in a reference frame in free fall, are the same as in the absence of gravity

In practice: For a sufficiently small “elevator box”, over sufficiently little time, tidal effects will occur below the detection limit.

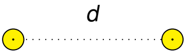
Spacetime picture of spheres sitting in space



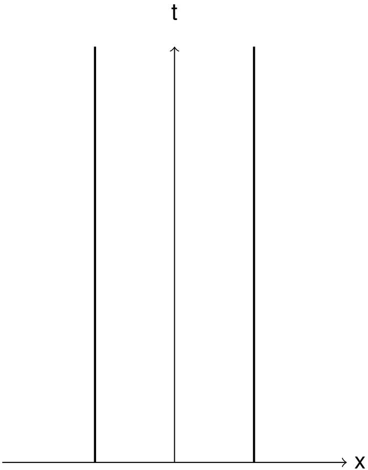
How does this look in a spacetime diagram?



Spacetime picture of spheres sitting in space

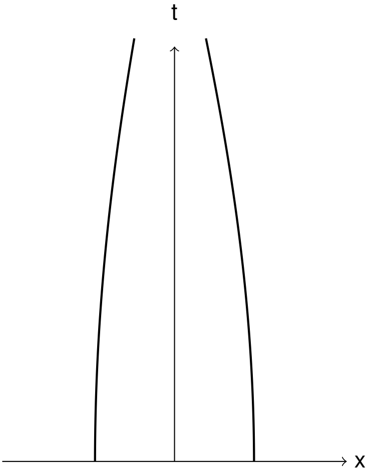


Spacetime diagram
(simplified version)



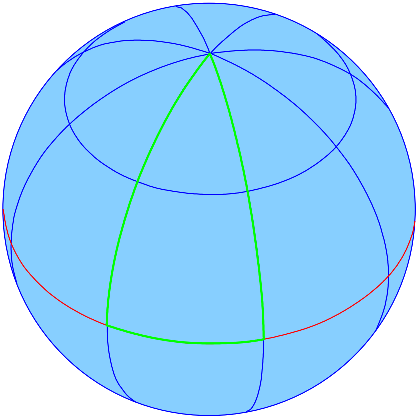
Spacetime picture of tidal forces during free fall

Initially, the world lines are parallel. Then they converge!



. . . so what are free-fall world-lines?

We have seen that, in the presence of gravity, the free-fall worldlines will converge or diverge - even if they are initially parallel. If you know your geometry, that should remind you of something:



Suggests analogy: curved surfaces!

No forces: straight worldlines	Straightest-possible curves on plane: straight lines
Free fall: worldlines can converge	on a curved surface: straightest-possible curves can converge
Equivalence principle	on infinitesimal scales, curved surface looks flat

Mechanics by procrastination I

Let's reformulate mechanics a bit:

1. Free particles travel along straight lines, at constant speed
2. The worldlines of free particles are straight lines in spacetime
3. [hold on]

Mechanics by procrastination II

A (seemingly) different question: How much time passes along a world-line? This time is called *proper time*, often denoted by τ — time as shown by a co-moving clock.

Assume we're in an inertial system S. For simplicity, all motion is along the x-axis, the particle's instantaneous speed is v, and its trajectory in S: $x = vt$.

Introduce (currently) co-moving inertial system S'. What time passes in S' as $\mathrm{d}t$ passes in S?

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \frac{(1 - v^2/c^2) \cdot t}{\sqrt{1 - v^2/c^2}} = \sqrt{1 - v^2/c^2} \cdot t$$

Mechanics by procrastination III

Result: time dilation,

$$\mathrm{d}\tau^2 = \mathrm{d}t'^2 = (1 - v^2/c^2) \cdot \mathrm{d}t^2$$

— between the same two events, $\mathrm{d}t'$ is larger.
Can be re-written using $\mathrm{d}x = v\mathrm{d}t$:

$$\mathrm{d}\tau^2 = \mathrm{d}t'^2 = \mathrm{d}t^2 - \mathrm{d}x^2/c^2.$$

Exercise:

Using the Euler-Lagrange equations, show that the extremal trajectories (in the x-t-plane) are straight lines $x = vt + t_0$. For a special case, show that along those trajectories, *maximum proper time* passes.

Mechanics by procrastination IV

Principle of procrastination:

Free particles will move in such a way that *maximum proper time* passes between each two events on their worldline.

So what if we tweak the situation a bit?

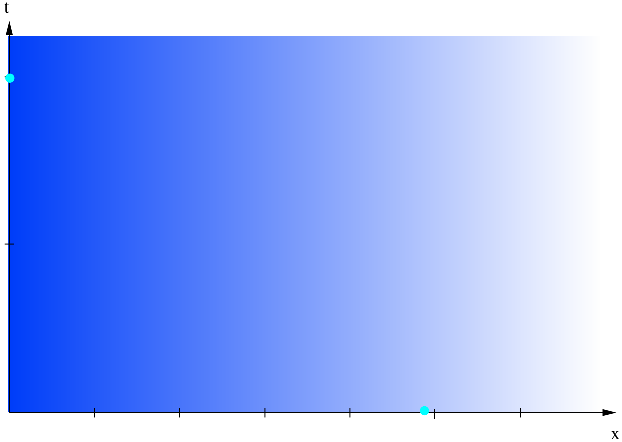
$$\mathrm{d}\tau^2 = (1 + f(x))\,\mathrm{d}t^2 - \mathrm{d}x^2/c^2.$$

Time running at different rates depending on position — how do particles move?

Mechanics by procrastination V

Linear case (simplest Ansatz):

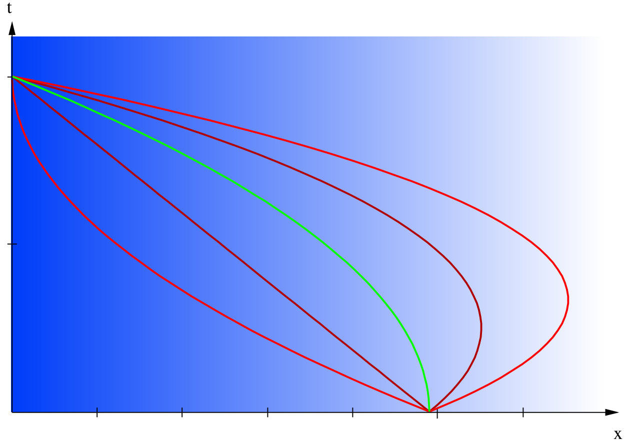
$$\mathrm{d}\tau^2 = (1 + d\cdot x)\,\mathrm{d}t^2 - \mathrm{d}x^2/c^2.$$



Mechanics by procrastination V

Linear case (simplest Ansatz):

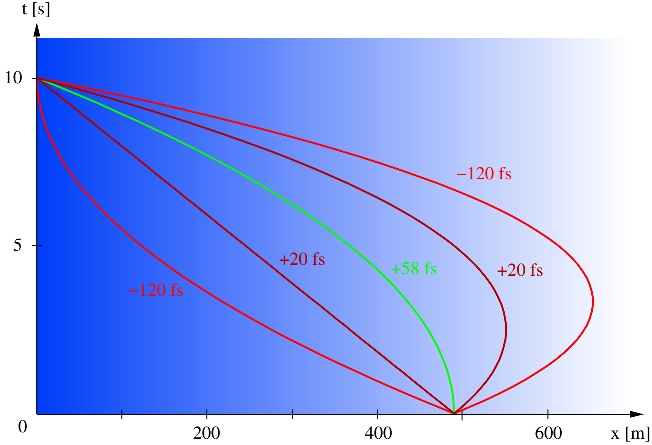
$$\mathrm{d}\tau^2 = (1 + d \cdot x)\,\mathrm{d}t^2 - \mathrm{d}x^2/c^2.$$



Mechanics by procrastination VI

With proper proportionality factor, reproduce simple fall:

$$d\tau^2 = (1 + 2 \cdot g/c^2 \cdot x)\, dt^2 - dx^2/c^2.$$



Mechanics by procrastination, summary

Let's reformulate mechanics a bit:

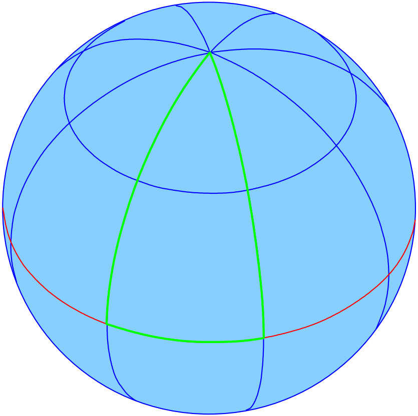
1. Free particles travel along straight lines, at constant speed
2. The worldlines of free particles are straight lines in spacetime
3. The worldlines of free particles are worldlines of maximal proper time
4. The worldlines of particles in free fall can *also* be described as worldlines of maximal proper time

Summary: Preparation for general relativity

- Free fall replaces free motion
- Extremal worldlines replace straight worldlines
- Analogies with distorted/curved surfaces
- “Time distortion” reproduces simple, Newtonian gravity

...time to take a closer look at curved spaces!

A simple curved surface: the sphere



[more info on the blackboard]

Introducing general coordinates

The three-fold use of coordinates:

- Labels to identify points
- Encode closeness (topological space)
- Encode distances (space with metric, e.g. $l = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$)

As we generalize from simple, Euclidean space, we will have to look at these roles in turn!

Coordinates on a wavy surface

Let's begin in two dimensions: with a smooth, but wavy, hilly surface ("Buckelpiste"):

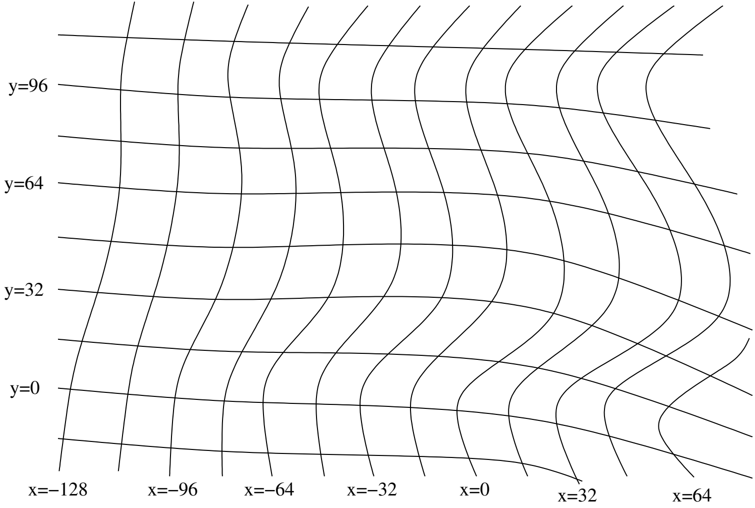
Coordinates on a wavy surface

Even better: Imagine that the surface is pure, smooth rock.

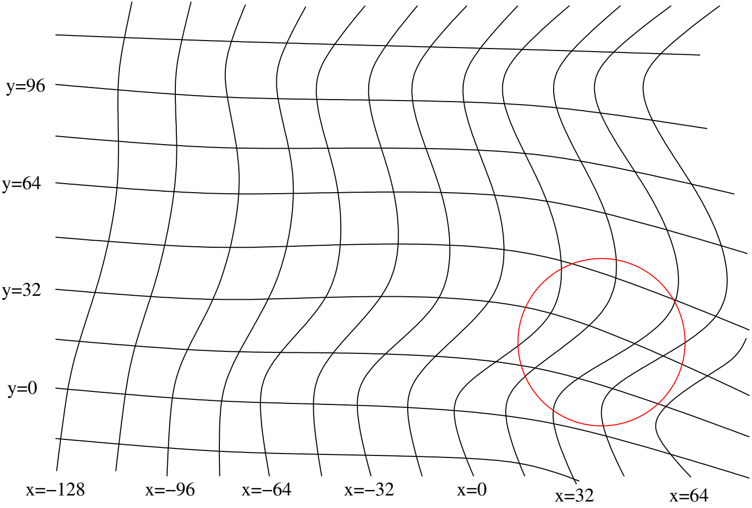
Now, put coordinate lines on it. (Purpose, for a start: Identifying different points.)

The lines are going to be curvy and wavy.

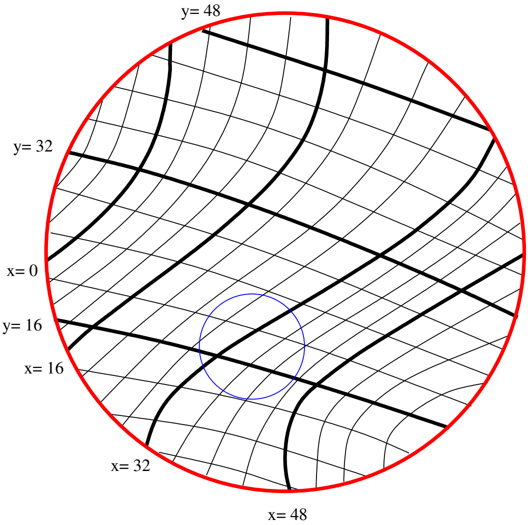
Coordinates on a wavy surface



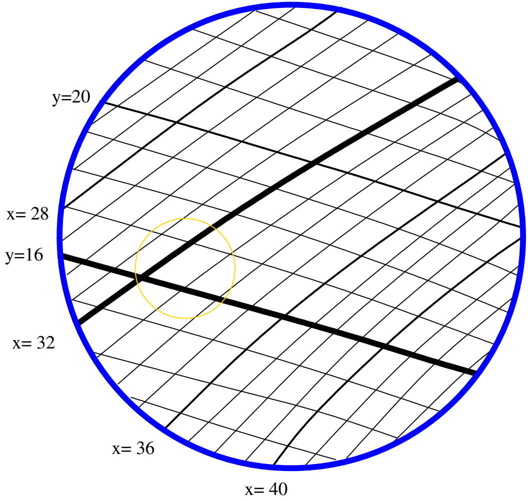
Coordinates on a wavy surface



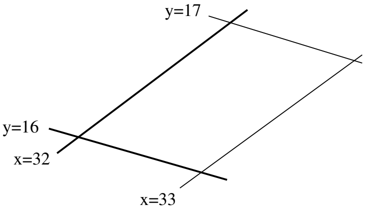
Coordinates on a wavy surface



Coordinates on a wavy surface

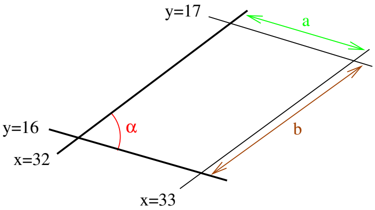


Coordinates on a wavy surface



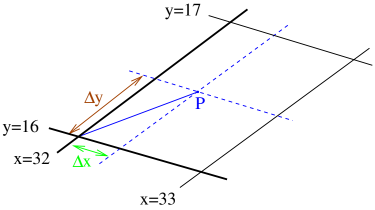
This if fairly simple - a parallelogram!

Coordinates on a wavy surface



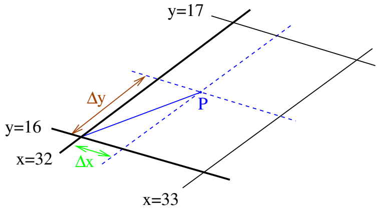
Assume an isometric view (straight down onto the plane): read off 3 parameters!

Coordinates on a wavy surface



What's the length of the blue line between $(32, 16)$ and P?

Coordinates on a wavy surface



$\vec{P} = (b\,\Delta y)\,\vec{u}_y + (a\,\Delta x)\,\vec{u}_x$ where $\vec{u}_x \cdot \vec{u}_y = \cos\alpha$ means that

$$|\vec{P}|^2 = a^2\,\Delta x^2 + 2ab\,\cos\alpha\,\Delta x\Delta y + b^2\,\Delta y^2.$$

With this modification, our coordinates can be used to measure lengths!

The metric

$$\Delta s^2 \;=\; |\vec{P}|^2 = a^2\,\Delta x^2 + 2ab\,\cos\alpha\,\Delta x\Delta y + b^2\,\Delta y^2$$

Three independent parameters. Let's rename them:

$$\Delta s^2 \;=\; g_{11}\,\Delta x^2 + 2g_{12}\,\Delta x\Delta y + g_{22}\,\Delta y^2.$$

We can write this in matrix form:

$$\Delta s^2 \;=\; (\Delta x, \Delta y)\cdot\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\cdot\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}.$$

(this is a *quadratic form*)

The metric

$$\Delta s^2 = (\Delta x, \Delta y) \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Things to take into account:

- the g_{ij} are location-dependent – write as $g_{ij}(x)$ where, in this case, x stands for all the coordinates
- this is true only in an infinitesimal neighbourhood, so let's write $\mathrm{d}x$ instead of Δx etc.

$$\mathrm{d}s^2 = (\mathrm{d}x, \mathrm{d}y) \cdot \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{12}(x) & g_{22}(x) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Some new notation I

$$\mathrm{d}s^2 = (\mathrm{d}x, \mathrm{d}y) \cdot \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{12}(x) & g_{22}(x) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Numbering coordinates as $\mathrm{d}x^1 \equiv \mathrm{d}x$ and $\mathrm{d}x^2 \equiv \mathrm{d}y$ (do not confuse with exponents!)

Introducing indices such as $i = 1, 2$:

$$\mathrm{d}s^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x)\, \mathrm{d}x^i \mathrm{d}x^j.$$

where we need to introduce $g_{21} \equiv g_{12}$ to keep things simple.

Some new notation II

$$\mathrm{d}s^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j.$$

Introduce the *Einstein summation convention*: Indices that occur twice (once up, once down) in the same product are automatically summed over!

$$\mathrm{d}s^2 = g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j.$$

g_{ij} are the components of the *metric* or *line element* — the central element of the geometry of generalized surfaces and spaces!

Form is valid for other number of dimensions, too. What we do from now on can be generalized!

Special case: the Euclidean plane

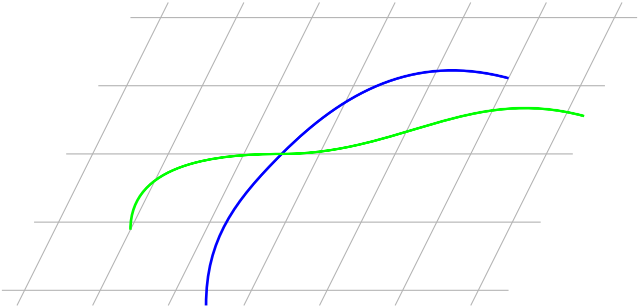
$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So that

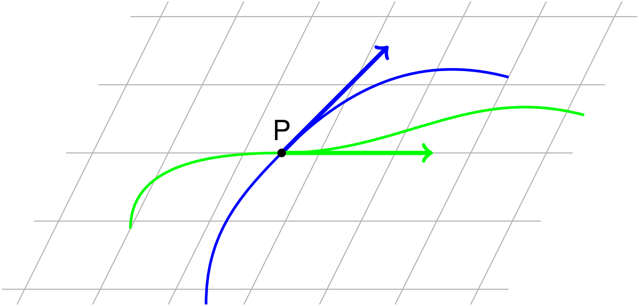
$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2.$$

Curves

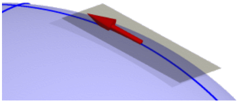
Curves: functions $c(\lambda)$ onto the surface that depend on a parameter $c : \mathbb{R} \to \mathbb{R}^2$. Components: $c^i(\lambda)$ give the coordinates of the curve's location for each parameter value.

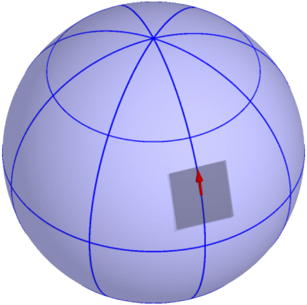


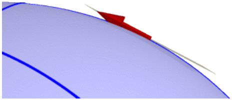
Defining directions



Defining directions







Defining directions

Scalar function on surface, e.g. temperature: $T:\mathbb{R}^2\to\mathbb{R}$.

Derivative to show how temperature changes along a curve $c(\lambda)$:

$$\frac{\mathrm{d}T(c(\lambda))}{\mathrm{d}\lambda}=\frac{\mathrm{d}c^i}{\mathrm{d}\lambda}\,\frac{\partial T(x)}{\partial x^i}$$

The expression

$$v^i=\frac{\mathrm{d}c^i}{\mathrm{d}\lambda}$$

encodes the direction of our curve at each particular parameter value λ — this is a generalized *vector*.

Using the metric to define norm and scalar product

$$|v|^2 \equiv g(v,v) = g_{ij}\, v^i\, v^j.$$

In the Euclidean plane, this reduces to the familiar formula

$$|v|^2 = v_x^2 + v_y^2.$$

$$\vec{v} \cdot \vec{w} = g(v,w) = g_{ij}\, v^i\, w^j$$

In Euclidean plane, this reduces to

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y.$$

Define angle ϕ between two vectors:

$$\cos\phi = \frac{\vec{v} \cdot \vec{w}}{|v| \cdot |w|}.$$

Volume elements

In *n*-dimensional Euclidean space:

$$\mathrm{d}V = \mathrm{d}x^1 \cdot \mathrm{d}x^2 \cdots \mathrm{d}x^n.$$

If coordinates are orthogonal, but not normalized: rescale each $\mathrm{d}x^i$ by $\sqrt{g_{ii}}$ to get proper length; result is

$$\mathrm{d}V = \sqrt{g_{11} \cdot g_{22} \cdots g_{nn}}\, \mathrm{d}x^1 \cdot \mathrm{d}x^2 \cdots \mathrm{d}x^n.$$

More general (won't prove it; won't use it):

$$\mathrm{d}V = \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \cdot \mathrm{d}x^2 \cdots \mathrm{d}x^n.$$

Different coordinate choices

Our choice of coordinates was fairly arbitrary. So what happens is we choose a different coordinate system?

$$\begin{aligned} x' &= x'(x,y) \\ y' &= y'(x,y) \end{aligned}$$

Small coordinate changes are governed by the chain rule:

$$\mathrm{d}x' = \frac{\partial x'}{\partial x}\cdot \mathrm{d}x + \frac{\partial x'}{\partial y}\mathrm{d}y$$

and

$$\mathrm{d}y' = \frac{\partial y'}{\partial x}\cdot \mathrm{d}x + \frac{\partial y'}{\partial y}\mathrm{d}y$$

or

$$\mathrm{d}x'^i = \frac{\partial x'^i}{\partial x^j}\cdot \mathrm{d}x^j \quad \text{– note lower } j \text{ index! Convention!}$$

Coordinate transformations

$$\frac{\partial x'^i}{\partial x^j}$$

is the corresponding *Jacobi matrix* for a coordinate transformation.

To get back from x' to x, use its (matrix) inverse,

$$\frac{\partial x^i}{\partial x'^j},$$

so that

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial x'^j} \cdot \mathrm{d}x'^j.$$

Coordinate transformations and the metric

Our choice of coordinates was completely arbitrary. Distance, on the other hand, isn't: If we pick two (infinitesimally close) points P_1 and P_2, and use two different coordinate systems and their metric coefficients to describe their distance, we must get the same result:

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \stackrel{!}{=} g'_{ij}\,\mathrm{d}x'^i\,\mathrm{d}x'^j,$$

where all the coordinates and metric coefficients are evaluated at, say, P_1 and the coordinate differences are those between P_2 and P_1.

Coordinate transformations and the metric

Inserting the transformation formula (Jacobi matrix) for $x \to x'$, it follows that

$$\begin{aligned} g'_{ij}\,\mathrm{d}x'^i\,\mathrm{d}x'^j &= g'_{ij}\left(\frac{\partial x'^i}{\partial x^k}\right)\left(\frac{\partial x'^j}{\partial x^l}\right)\mathrm{d}x^k\mathrm{d}x^l = g'_{kl}\left(\frac{\partial x'^k}{\partial x^i}\right)\left(\frac{\partial x'^l}{\partial x^j}\right)\mathrm{d}x^i\mathrm{d}x^j \\ &\stackrel{!}{=} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \end{aligned}$$

so that

$$g_{ij} = g'_{kl}\left(\frac{\partial x'^k}{\partial x^i}\right)\left(\frac{\partial x'^l}{\partial x^j}\right).$$

Coordinate transformations and invariance

This is crucial! We have introduced great arbitrariness (the freedom to choose coordinates), but some things are *invariant* (e.g. distances), and from this we can deduce simple (linear) transformation behaviour.

Coordinate changes and invariance

On second thoughts, this shouldn't be so unfamiliar.

Compare this with classical mechanics and (fixed-angle) rotated coordinate systems: For

$$\vec{F} = m\vec{a}$$

the LHS and RHS transform in the same way if you rotate the coordinate system, $\vec{F} \mapsto R\vec{F}$ and $\vec{a} \mapsto R\vec{a}$ so the equation remains valid!

The same is true for our coordinate transformations (with Jacobi matrix)!

Covariance and contravariance

Contravariant vector: An object whose components transform as

$$v^i \mapsto v'^i = \left(\frac{\partial x'^i}{\partial x^j}\right) \, v^j$$

Example: $\mathrm{d}x^i$.

Covariant vector: An object whose components transform as

$$v_i \mapsto v'_i = \left(\frac{\partial x^j}{\partial x'^i}\right) \, v_j$$

Example:

$$\frac{\partial \phi(x)}{\partial x^i}$$

Invariant combinations and tensors

Example:

$$g(v,v) \equiv g_{ij}v^i v^j = g'_{ij}\, v'^i\, v'^j$$

— if v^i is a contravariant vector, this is an invariant combination!

Define generalized linear functions:

$$T(\underbrace{u, v, ..., w}_{m \text{ contrav.}}, \underbrace{\omega, \xi, \ldots, \chi}_{n \text{ cov.}}) = T_{ij\cdots k}{}^{pq\cdots r}\, u^i\, v^j\, \cdots\, w^k\, \omega_p\, \xi_q\, \cdots \chi_r$$

T is *tensor of type (n,m)* (that is, with *n* contravariant indices and *m* covariant indices).

Raising and lowering indices

Define contravariant metric as the tensor of type (2,0) g^{ij} so that

$$g^{ij} g_{jk} = \delta^i_k$$

where $\delta^i_k = 1$ if $i = k$ and zero otherwise.

Use g^{ij} to raise indices: If v_i is a covariant vector, then

$$v^j = g^{ji}\, v_i$$

are the components of a contravariant vector — in the other direction, indices can be lowered with g_{ij}.

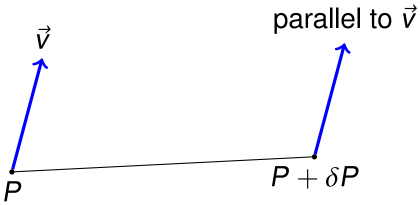
A side-remark for the mathematically minded

Mathematically rigorous definitions are slightly different:

- surfaces and their generalizations as *manifolds* covered with coordinate patches
- vectors as *directional derivatives*, defining directions at each point
- 1-forms as *linear functions* of vectors to the reals
- tensors of rank (p,q) as *multilinear functions* with *q* vector and *p* 1-form arguments

Parallel transport

We have seen that vectors at different points of the surface live on different tangent planes. So how can we compare different vectors?



Evidently, we need a map from one point to another.

Parallel transport

Simplest Ansatz: the vector $\vec{v}_P$ at $x + \delta x$ that is parallel to $\vec{v}$ at x has components

$$v_P^i(x + \delta x) = v^i(x) - \Gamma^i_{jk}(x)\, v^j(x)\, \delta x^k.$$

The Γ^i_{jk} are called *connection coefficients*. They define a *connection*, a notion of *parallel transport*.

Covariant derivative

With the help of the connection, we can define the *covariant derivative* to track changes of a vector from one location to another:

$$(\nabla_i v)^j(x) \equiv \lim_{\delta x^i \to 0} \frac{v^j(x+\delta x) - v_P^j(x+\delta x)}{\delta x^i}.$$

By the definition of $v_P^j(x+\delta x)$, this is

$$(\nabla_i v)^j(x) = \lim_{\delta x^i \to 0} \frac{v^j(x+\delta x) - v^j(x)}{\delta x^i} + \Gamma^j_{ki} v^k(x) = \frac{\partial v^j(x)}{\partial x^i} + \Gamma^j_{ki} v^k(x).$$

Covariant derivative and Leibniz rule

For scalar functions, define

$$\nabla_i \phi = \frac{\partial \phi}{\partial x^i} \equiv \partial_i \phi.$$

Also, demand that the Leibniz rule holds — for instance,

$$\nabla_i (v^j\,\omega_j) = \partial_i (v^j\,\omega_j) = (\nabla_i v)^j \omega_j + v^j (\nabla_i \omega)_j$$

This defines the action of the covariant derivative on one-form components as

$$(\nabla_i \omega)_j = \partial_i \omega_j - \Gamma^k_{ji} \omega_k.$$

Allows for generalizations to all types of tensors.

Metric connection 1/3

But in addition to the connection (which defines parallel transport), we also have the metric (which defines lengths of vectors and angles between vectors). We need to make sure those two are compatible!

Compare at points x and $x + \delta x$ vector fields v, w with

$$\delta x^i (\nabla_i v)^j = 0 \quad \text{and} \quad \delta x^i (\nabla_i w)^j = 0,$$

in other words: $v_P(x + \delta x) = v(x)$ and the same for w.

Then the angle between $v_P(x + \delta x)$ and $w_P(x + \delta x)$ should be the same as between $v(x)$ and $w(x)$,

$$g(v, w)\big|_x = g(v, w)\big|_{x+\delta x}$$

Metric connection 2/3

$g(v, w)$ is a scalar, so this translates to

$$\delta x^i \partial_i (g(v, w)) = \delta x^i \nabla_i g(v, w) = 0.$$

Using the Leibniz rule and $\delta x^i (\nabla_i v)^j = 0$ (same for w):

$$\begin{aligned} 0 &= \delta x^i \nabla_i g(v, w) = \delta x^i \left[g_{jk}\, v^j\, (\nabla_i w)^k + g_{jk}\, (\nabla_i v)^j\, w^k + (\nabla_i g)_{jk}\, v^j\, w^k \right] \\ &= (\nabla_i g)_{jk}\, \delta x^i\, v^j\, w^k. \end{aligned}$$

We are free to choose sufficiently many different, linearly independent δx^i; the same holds, separately, for v^j and for w^k, to deduce

$$(\nabla_i g)_{jk} = 0.$$

Metric connection 3/3

Written in components,

$$0 = (\nabla_i g)_{jk} = 0 = \partial_i g_{jk} - \Gamma^l_{ji} g_{lk} - \Gamma^l_{ki} g_{jl}.$$

Direct calculation shows that

$$\frac{1}{2} g^{lm} (\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}) = \Gamma^l_{jk}$$

In other words: our compatibility condition completely determines the connection coefficients!

The result is called the *metric connection* or *Levi-Civita connection*.

How the Levi-Civita connection transforms

From the definition, it follows that

$$\Gamma'^{i}_{jk} = \frac{\partial x'^i}{\partial x^l}\frac{\partial x^m}{\partial x'^j}\frac{\partial x^n}{\partial x'^k}\Gamma^l_{mn} - \frac{\partial x^l}{\partial x'^j}\frac{\partial x^m}{\partial x'^k}\frac{\partial^2 x'^i}{\partial x^l \partial x^m}.$$

Evidently not a tensor!

However, transforms just the right way for

$$(\nabla'_i v')^j = \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^i}{\partial x^l}(\nabla_k v')^l$$

to transform like a tensor.

Geodesics

Geodesics: generalization of *straight lines*: parametrizing by arc length λ, curve $x(\lambda)$ with

$$\dot{x}^i \equiv \frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda},$$

straightest-possible means no change of the tangent vector as we move along the curve,

$$\dot{x}^i(\nabla_i \dot{x})^j = 0 \quad = \frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda}\partial_i \dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}^k$$

$$\text{and} \quad \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \quad \text{(via chain rule)}$$

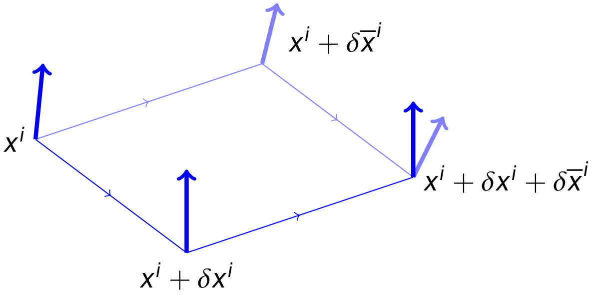
Same equation results from Euler-Langrange equations for

$$\delta \int \sqrt{g(\dot{x},\dot{x})}\mathrm{d}\lambda = 0;$$

$\Rightarrow$ *extremal* (in ordinary space: shortest possible lines).

Curvature

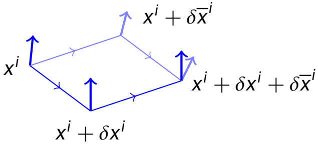
What if we parallel-transport a vector from P to Q along two different ways?



Example: Parallel transport on Earth!

Curvature

Next: Put this into a formula!



Parallel transport:

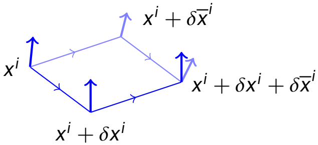
$$v^i(x+\delta\overline{x}) = v^i(x) - \Gamma^i_{jk}(x)\, v^j(x)\, \delta\overline{x}^k$$

so (light-blue path)

$$v^i(x+\delta\overline{x}+\delta x) = v^i(x+\delta\overline{x}) - \Gamma^i_{jk}(x+\delta\overline{x})\, v^j(x+\delta\overline{x})\, \delta x^k$$

Curvature

Inserting expressions:



$$
\begin{aligned}
v^i(x+\delta\bar{x}+\delta x) \quad &= \quad v^i(x) - \Gamma^i_{jk}(x)v^j(x)\delta\bar{x}^k \\
&\quad -\left[\Gamma^i_{jk}(x) + \delta\bar{x}^l\partial_l\Gamma^i_{jk}(x)\right)\Big]\left[v^j(x) - \Gamma^j_{ml}(x)v^m(x)\delta\bar{x}^l\right]\delta x^k
\end{aligned}
$$

so that

$$v^i(x+\delta\bar{x}+\delta x) - v^i(x+\delta x+\delta\bar{x}) = R^i{}_{jkl}(x)v^j(x)\delta x^k\delta\bar{x}^l$$

with

$$R^i{}_{jkl}(x) = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}$$

Curvature

$$R^i{}_{jkl}(x) = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}$$

is called **Riemann curvature tensor** Warning: Overall sign conventional!

(Check transformation behaviour to see it's really a tensor!)

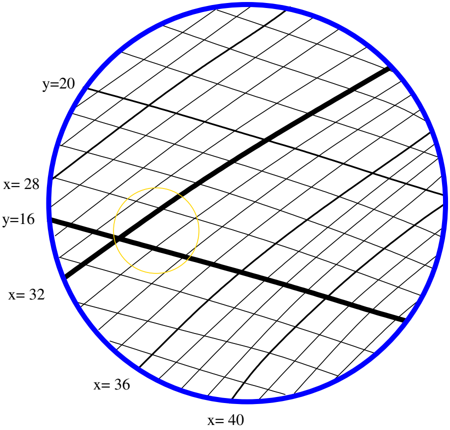
Various contractions/combinations:

$R_{jl} \equiv R^i{}_{jil}$ is the **Ricci tensor** Sign conventional!

$R \equiv g^{jl} Ric_{jl}$ is the **Ricci scalar** or **scalar curvature**

$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ is the **Einstein tensor**

Infinitesimal local flatness 1/2



Coordinate transformations possible ⇒ think of our smooth rock covered by a rubber sheet!

Infinitesimal local flatness 2/2

General recipe: Coordinate change at point P,

$$x'^i = M^i{}_j x^j + \frac{1}{2} N^i{}_{jk} x^j x^k + O(x^3)$$

and

$$x^i = M'^i{}_j x'^j + O(x^2)$$

$$\Rightarrow g'_{ij} = g_{kl} M'^k{}_i \, M'^l{}_j + O(x)$$

— can be solved for $g'_{ij} = \delta_{ij}$ with the right M'.

$$\Gamma'^i_{jk} = M'^m{}_j \, M'^n{}_k (M^i{}_l \, \Gamma^l_{mn} - N^i{}_{mn})$$

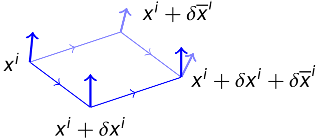
Can be solved for $\Gamma'^i_{jk} = 0$ by choosing $N^i{}_{mn} = M^i{}_l \, \Gamma^l_{mn}$.

Infinitesimal local flatness and curvature

Apparently, we can always find a coordinate system for which $g_{ij} = \delta_{ij}$ and $\Gamma^i{}_{jk} = 0$ at some point P!

What can keep us from transporting the simple basis vectors from P to other points and establish a Euclidean coordinate system *everywhere*?

No unique way of transporting!



Curvature, $R^i{}_{jkl} \neq 0$, keeps us from having a flat coordinate system everywhere!

Special relativity revisited

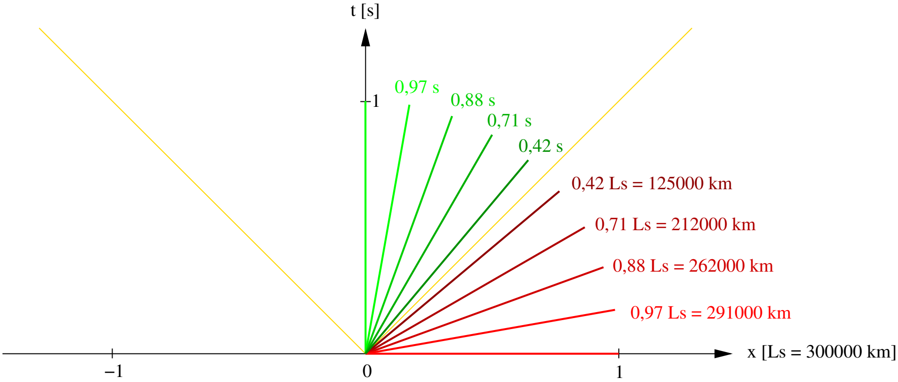
One can define a metric in special relativity, but it doesn't look like the ones we've encountered. This is the *Minkowski metric*:

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = \mathrm{d}\vec{x}^2 - c^2\mathrm{d}t^2.$$

This is invariant under Lorentz transformations!

But what does it mean?

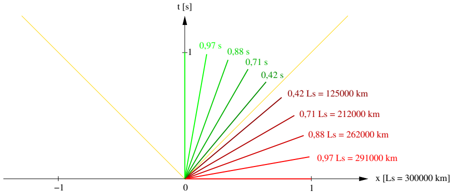
The meaning of the SR metric



$$ds^2 = -c^2 d\tau^2 = d\vec{x}^2 - c^2 dt^2.$$

The meaning of the SR metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = \mathrm{d}\vec{x}^2 - c^2\mathrm{d}t^2.$$



- **timelike**, $\mathrm{d}s^2 < 0$: possible worldlines of $(m > 0)$ particles
- **lightlike**, $\mathrm{d}s^2 = 0$: light-cone
- **spacelike**, $\mathrm{d}s^2 > 0$: possible spatial distance

Some (index and other) conventions

Conventions differ — some write a, b, c indices, some α, β, γ; some have their indices run from 0 to 3 (time is 0), some from 1 to 4 (time is 4).

We choose greek indices starting in the middle of the alphabet, $\mu, \nu, \rho, \ldots$, for spacetime indices running from 0 to 3.

If only the spatial coordinates 1,2,3 are involved, we use indices $i, j, k, \ldots$.

We choose the convention $c = 1$, and the metric so that its spatial coordinates are positive ("mostly plus")

The Lorentz metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 \equiv \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

with

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

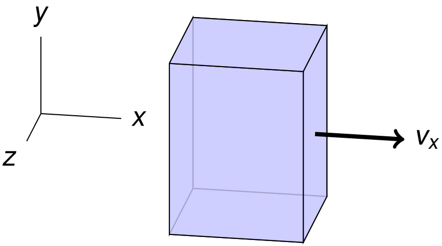
— up to sign, this is Euclidean. But the sign makes all the difference!

Continua and fluids

Continuum approximation in classical physics: collection of particles too numerous to track individually; introduce *collective properties* such as particle number density, energy/momentum densities, pressure, viscosity etc.

Fluid: continuum that has no internal rigidity or form stability — where a solid body reacts to deformation with e.g. shear forces, a fluid will just flow.

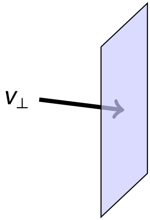
How to describe space-filling matter



- Find *momentarily co-moving reference frame* (MCRF)
- Define rest density for interesting quantities
- Transform into moving system

Example: Number density

- In MCRF: number density n
- Reference frame relative to which MCRF moves at speed $\vec{v}$: $n\gamma(v)$ (Lorentz contraction)
- This is not a scalar! Complete to relativistic four-vector $N^\mu = n\gamma(v)(1, \vec{v}) = nu^\mu$
- So what are the $n\,\gamma(v)\,\vec{v}$? Fluxes:



Flux f through surface (say, in x-direction): Number of particles flowing through is $N = f \cdot \Delta y \Delta z \Delta t$, with $f = n\gamma(v)\,v_\perp$.

- Density and fluxes are linked – density is “flow in the time direction”

Energy-momentum flow

For particles, other quantities can flow, as well. Notably four-momentum,

$$p^\mu = m\,u^\mu.$$

with m the rest mass. If the rest energy density for system of particles is $\rho = m\,n$, the flow of energy and momentum is described by

$$T^{\mu\nu} = \rho\,u^\mu u^\nu,$$

namely four-momentum flow in each four directions.

Energy-momentum tensor

…also known as *stress-energy tensor*.

General form:
$T^{\mu\nu} =$ flux of p^{μ} in the ν-direction.

T^{00}	energy density
T^{0i}	energy flux in *i*-direction
T^{i0}	*i*-component of momentum density
T^{ij}	*i*-momentum flux in *j*-direction

Energy-momentum conservation 1/3

Within each little volume element, energy or momentum only change when net energy or momentum flow in! Example: energy conservation at a spacetime point P

Within a time interval Δt, in small (square) volume $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z = \Delta x^1 \cdot \Delta x^2 \cdot \Delta x^3$: *Net outflow* is

$$
\begin{aligned}
& \left[T^{01}(P+\Delta x) - T^{01}(P)\right] \Delta y \cdot \Delta z \cdot \Delta t \\
+ \quad & \left[T^{02}(P+\Delta y) - T^{02}(P)\right] \Delta x \cdot \Delta z \cdot \Delta t \\
+ \quad & \left[T^{03}(P+\Delta z) - T^{03}(P)\right] \Delta x \cdot \Delta y \cdot \Delta t.
\end{aligned}
$$

Energy-momentum conservation 2/3

Re-write the net outflow (using e.g. $x^0 = t$):

$$\left[\frac{T^{0i}(P + \Delta x^i) - T^{0i}(P)}{\Delta x^i}\right] \Delta V \cdot \Delta x^0$$

Net change of energy inside the volume is:

$$\left[T^{00}(P + \Delta t) - T^{00}(P)\right] \Delta V = \left[\frac{T^{00}(P + \Delta x^0) - T^{00}(P)}{\Delta x^0}\right] \Delta V \cdot \Delta x^0$$

Equate the two, divide by $\Delta V \cdot \Delta x^0$, take infinitesimal limit:

$$\frac{\partial}{\partial x^\mu} T^{0\mu} = 0.$$

Energy-momentum conservation 3/3

Same argument works for momentum components!

General **energy conservation law**:

$$\frac{\partial}{\partial x^\mu} T^{\nu\mu} \equiv \partial_\mu T^{\nu\mu} = 0.$$

Energy-momentum for a perfect fluid 1/2

What if, in the MCRF, particle energy is more than their rest energy?

Particles moving randomly, as in a gas or liquid (that is, a fluid)?

Then, in rest frame,

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

…why? Think of liquid element, no net flow. T^{ij} would be *shear forces*. T^{i0} would mean net momentum. T^{0i} would mean flow of energy. All that remains is *pressure*, that is, random momentum exchange outward.

Energy-momentum for a perfect fluid 2/2

$T^{\mu\nu}$ written in terms of tensors:

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu}.$$

…for our examples so far: $T^{\mu\nu} = T^{\nu\mu}$. This is true in general; otherwise there would be a size-dependent torque on each small volume element (cf. Schutz, p. 103).

Relativistic limit of energy-momentum tensor

Once rest mass can be neglected:

$$E^2 = \vec{p}^2 + m^2 \quad \rightarrow \quad E = |\vec{p}|.$$

Model: Photons bouncing back and forth between perfect mirrors, distance L, in $\Delta x\,\Delta y\,\Delta z$ (reality: photon exchange) — similar to gas pressure in statistical mechanics; after averaging out angle-of-incidence,

$$p = \frac{1}{3}\,\rho.$$

Simple (interesting) perfect fluids

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu}$$

with special cases dictated by the (linear) equations of state
$p = w\rho$

1. **Dust**: $w = 0 \Rightarrow p = 0$
2. **Radiation**: $w = \frac{1}{3} \Rightarrow p = \frac{1}{3}\rho$
3. **Scalar field** (or Dark Energy): $w = -1 \Rightarrow p = -\rho$

Bringing it all together

Correspondence physics $\Rightarrow$ (curved) spacetime:

- Free fall worldlines $\Rightarrow$ geodesics
- tidal component of gravity $\Rightarrow$ curvature tensor
- Principle of equivalence $\Rightarrow$ local flatness (SR metric!)

In particular, the last item is a guideline for formulating physics in general relativity: the laws must reduce to those of special relativity in a locally flat frame! This is **Einstein's equivalence principle**.

One consequence: $\mathrm{d}s^2 = 0$ is light propagation in general relativity, as well!

Transition from special to general

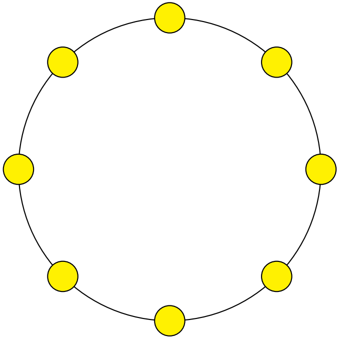
Implementing “physics by equivalence principle”: Use the laws formulated for special relativity and:

- whenever there is a metric term $\eta_{\mu\nu}$, generalize to $g_{\mu\nu}$
- whenever there is a partial derivative ∂_μ, generalize to ∇_μ

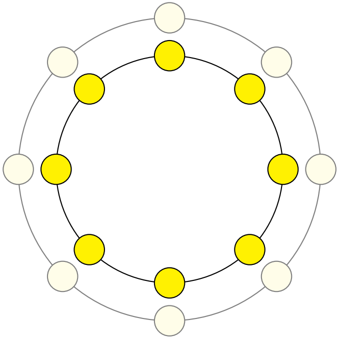
This automatically guarantees *general coordinate invariance* of equations — that is why it’s general relativity, not special relativity!

Still leaves the question: What about gravity?

Which curvature tensor?



$\Rightarrow$



This is what the **Einstein tensor** does: converge geodesics!

What causes curvature?

Tidal component of gravity must somehow be produced by mass (=energy, and since we need a vector or tensor, by momentum as well).

A tensor of the same type as Einstein tensor $G_{\mu\nu}$ is the **energy-momentum tensor**

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

This has the proper Newtonian limit and implements Newtonian gravity via $g_{00}(\vec{x})$ — position-dependent time!

Energy conservation

We have $(\nabla_\mu G)^{\mu\nu} = 0$. From Einstein's equations follows

$$(\nabla_\mu T)^{\mu\nu} = 0.$$

This is the general-relativistic generalization of energy-momentum conservation in special relativity, namely

$$\partial_\mu T^{\mu\nu} = 0.$$

The cosmological constant

Einstein derived his equations from mathematical desiderata: equations for $g_{\mu\nu}$ should be invariant under coordinate transformations, differential equations of no more than second order, and linear in the second derivatives (alternative to last: $\nabla T = 0$).

Turns out that his original equations aren't general enough. You can add another term, with a new parameter Λ, and obtain

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi G\; T_{\mu\nu}.$$

Λ is called the *cosmological constant* (historical reasons).

The cosmological constant and Dark Energy

Rewrite equation as

$$G_{\mu\nu} = 8\pi G\left[T_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu}\right].$$

Remember perfect fluid stress-energy tensor:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

– this is exactly the new term, for

$$\rho = -p = \frac{\Lambda}{8\pi G}.$$

Relativistic model-building

Coupled system of *Einstein equation*, *continuity equation* for stress-energy tensor, *equation of state* (specifying the properties of matter):

$$\underbrace{10}_{\text{metric}} + \underbrace{10}_{T_{\mu\nu}} = \underbrace{10}_{\text{Einstein eq.}} + \underbrace{4}_{\text{continuity}} + \underbrace{6}_{\text{e.o.s.}}$$

- General solutions: very messy ⇒ *numerical relativity*
- Exact solutions: simple models with symmetry
- Approximation (perturbation theory): e.g. gravitational waves

Each solution of general relativity is automatically a model universe!

The gravitational length scale

Consider a spherically symmetric vacuum solution of Einstein's field equations.

Birkhoff's theorem: There can only be one! [but with a parameter]

This is the **Schwarzschild solution** which, as was later shown, describes a *black hole*: A region of space from which nothing can escape (not even light).

Outside: Schwarzschild approaches Newtonian description. Characteristic length scale for where general relativity is dominant:

$$R_S = \frac{2GM}{c^2}$$

The gravitational length scale

$$R_S = \frac{2GM}{c^2}$$

Typical Schwarzschild radii:

$1M_\odot$: 3 km

$1M_\oplus$: 1 cm

Hoop conjecture: An object of mass M will collapse into a black hole if it is compressed in a way that you can pass a hoop with radius $R_S(M)$ around it in all orientations.

What we will need for cosmology

- We must find a metric to describe our cosmological model
- Use freedom of choosing coordinates to choose practical coordinates
- Properties related to matter content (energy-momentum tensor) by Einstein's equations
- Free-particle movement in that model: geodesics!
- Light propagation in that model: null geodesics $\mathrm{d}s^2 = 0$

Exact solutions

Exact solutions are, by necessity *simple model situations*.
Assumption: symmetries!

- Minkowski spacetime (empty)
- Schwarzschild solution (empty w/boundary: black hole)
- Kerr solution (rotating body: rotating bh, gravitomagnetism)
- Friedmann-Lemaître-Robertson-Walker (cosmology)

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